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## CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS III

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ABSTRACT. Let  $p > 3$  be a prime, and let  $R_p$  be the set of rational numbers whose denominator is coprime to  $p$ . Let  $\{P_n(x)\}$  be the Legendre polynomials. In this paper we mainly show that for  $m, n, t \in R_p$  with  $m \not\equiv 0 \pmod{p}$ ,

$$P_{[\frac{p}{6}]}(t) \equiv -\left(\frac{3}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3x + 2t}{p}\right) \pmod{p},$$

$$\left(\sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right)\right)^2 \equiv \left(\frac{-3m}{p}\right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{4m^3 + 27n^2}{12^3 \cdot 4m^3}\right)^k \pmod{p},$$

where  $\left(\frac{a}{p}\right)$  is the Legendre symbol and  $[x]$  is the greatest integer function. As an application we solve some conjectures of Z.W. Sun and the author concerning  $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} / m^k \pmod{p^2}$ , where  $m$  is an integer not divisible by  $p$ .

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### 1. Introduction.

Let  $\{P_n(x)\}$  be the Legendre polynomials given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \geq 1).$$

It is well known that (see [B, p. 151], [G, (3.132)-(3.133)])

$$(1.1) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n.$$

From (1.1) we see that

$$(1.2) \quad P_n(-x) = (-1)^n P_n(x), \quad P_{2m+1}(0) = 0 \quad \text{and} \quad P_{2m}(0) = \frac{(-1)^m}{2^{2m}} \binom{2m}{m}.$$

We also have the following formula due to Murphy ([G, (3.135)]):

$$(1.3) \quad P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k = \sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k.$$

We remark that  $\binom{n}{k}\binom{n+k}{k} = \binom{2k}{k}\binom{n+k}{2k}$ .

Let  $\mathbb{Z}$  be the set of integers, and for a prime  $p$  let  $R_p$  be the set of rational numbers whose denominator is coprime to  $p$ . Let  $[x]$  be the greatest integer not exceeding  $x$ , and let  $\left(\frac{a}{p}\right)$  be the Legendre symbol. In [S4-S6] the author showed that for any prime  $p > 3$  and  $t \in R_p$ ,

$$(1.4) \quad P_{\frac{p-1}{2}}(t) \equiv -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 3(t^2 + 3)x + 2t(t^2 - 9)}{p} \right) \pmod{p},$$

$$(1.5) \quad P_{[\frac{p}{4}]}(t) \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \pmod{p},$$

$$(1.6) \quad P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 + 3(4t - 5)x + 2(2t^2 - 14t + 11)}{p} \right) \pmod{p}.$$

In the paper, by using elementary and straightforward arguments we prove that

$$(1.7) \quad P_{[\frac{p}{6}]}(t) \equiv -\left(\frac{3}{p}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 3x + 2t}{p} \right) \pmod{p}.$$

Moreover, for  $m, n \in R_p$  with  $m \not\equiv 0 \pmod{p}$  we have

$$(1.8) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \equiv \begin{cases} -(-3m)^{\frac{p-1}{4}} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) \pmod{p} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

and

$$\left( \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \right)^2 \equiv \left( \frac{-3m}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left( \frac{4m^3 + 27n^2}{12^3 \cdot 4m^3} \right)^k \pmod{p}.$$

It is well known that (see for example [S2, pp.221-222]) the number of points on the curve  $y^2 = x^3 + mx + n$  over the field  $F_p$  with  $p$  elements is given by

$$\#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right).$$

For positive integers  $a, b$  and  $n$ , if  $n = ax^2 + by^2$  for some integers  $x$  and  $y$ , we briefly say that  $n = ax^2 + by^2$ . Recently the author's brother Zhi-Wei Sun[Su1,Su4] and the author[S4] posed some conjectures for  $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} / m^k$  modulo  $p^2$ , where  $p > 3$  is a prime and  $m \in \mathbb{Z}$  with  $p \nmid m$ . For example, Zhi-Wei Sun conjectured that ([Su4, Conjecture 2.8]) for any prime  $p > 3$ ,

$$(1.9) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-96)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = -1, \\ \left(\frac{-6}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1 \text{ and so } 4p = x^2 + 19y^2. \end{cases}$$

Using (1.8) and known character sums we determine  $P_{[\frac{p}{6}]}(x) \pmod{p}$  for 11 values of  $x$  (see Corollaries 2.1-2.11), and  $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} / m^k \pmod{p^2}$  for  $m = -640320^3, -5280^3, -960^3, -96^3, -32^3 - 15^3, 20^3, 66^3, 255^3, 54000, -12288000$ . Thus we solve some conjectures in [Su1, Su4] and [S4]. For example, we confirm (1.9) in the case  $(\frac{p}{19}) = -1$  and prove (1.9) when  $(\frac{p}{19}) = 1$  and the modulus is replaced by  $p$ .

Let  $p > 3$  be a prime. In the paper we also determine  $\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} / 864^k \pmod{p^2}$  and establish the general congruence

$$(1.10) \quad \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1 - 432x))^k \equiv \left( \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 \pmod{p^2},$$

and pose some conjectures on supercongruences.

## 2. Congruences for $P_{[\frac{p}{6}]}(x) \pmod{p}$ .

**Lemma 2.1.** *Let  $p$  be an odd prime. Then*

- (i)  $\binom{\frac{p-1}{2}}{k} \equiv \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p} \quad \text{for } k = 0, 1, \dots, \frac{p-1}{2},$
- (ii)  $\binom{\frac{p-1}{2} - k}{2k} \equiv \frac{\binom{6k}{3k} \binom{3k}{k}}{4^{2k} \binom{2k}{k}} \pmod{p} \quad \text{for } k = 0, 1, \dots, [\frac{p}{3}],$
- (iii)  $\binom{[\frac{p}{3}] + k}{2k} \equiv \frac{1}{(-27)^k} \binom{3k}{k} \pmod{p} \quad \text{for } p \neq 3 \text{ and } k = 0, 1, \dots, [\frac{p}{3}].$

Proof. For  $k \in \{0, 1, \dots, \frac{p-1}{2}\}$  we have  $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p}$ . Thus (i) holds. Now suppose  $k \in \{0, 1, \dots, [\frac{p}{3}]\}$ . It is clear that

$$\begin{aligned} \binom{\frac{p-1}{2} - k}{2k} &= \frac{\frac{p-1-2k}{2} \cdot \frac{p-3-2k}{2} \cdots \frac{p-(6k-1)}{2}}{(2k)!} \equiv \frac{(2k+1)(2k+3) \cdots (6k-1)}{(-2)^{2k} \cdot (2k)!} \\ &= \frac{(6k)!}{4^k (2k)!^2 (2(k+1))(2(k+2)) \cdots (2(3k))} = \frac{(6k)!}{4^k (2k)!^2 \cdot 2^{2k} \cdot \frac{(3k)!}{k!}} \\ &= \frac{\binom{6k}{3k} \binom{3k}{k}}{4^{2k} \binom{2k}{k}} \pmod{p}. \end{aligned}$$

Thus (ii) is true. (iii) was given by the author in [S4, the proof of Lemma 2.3]. The proof is now complete.

**Lemma 2.2.** *Let  $p > 3$  be a prime and  $k \in \{0, 1, \dots, [\frac{p}{12}]\}$ . Then*

$$\binom{[\frac{p}{6}]}{k} \binom{2[\frac{p}{6}] - 2k}{[\frac{p}{6}]} \equiv (-1)^{[\frac{p}{6}]} 3^{3k + \frac{1-(\frac{p}{3})}{2}} 4^{[\frac{p}{6}] - k} \binom{\frac{p-1}{2}}{[\frac{p}{3}] - k} \binom{\frac{p-(\frac{p}{3})}{6} + k}{3k + \frac{1-(\frac{p}{3})}{2}} \pmod{p}.$$

Proof. Using Lemma 2.1(i) we see that

$$\begin{aligned}
\binom{[\frac{p}{6}]}{k} \binom{2[\frac{p}{6}] - 2k}{[\frac{p}{6}]} &= \frac{(2[\frac{p}{6}] - 2k)!}{k!([\frac{p}{6}] - k)!([\frac{p}{6}] - 2k)!} = \binom{2([\frac{p}{6}] - k)}{[\frac{p}{6}] - k} \binom{[\frac{p}{6}] - k}{k} \\
&\equiv (-4)^{[\frac{p}{6}] - k} \binom{\frac{p-1}{2}}{[\frac{p}{6}] - k} \binom{[\frac{p}{6}] - k}{k} \\
&= (-4)^{[\frac{p}{6}] - k} \binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2} - k}{[\frac{p}{6}] - 2k} \\
&\equiv (-4)^{[\frac{p}{6}] - 2k} \binom{2k}{k} \frac{(\frac{p-1}{2} - k)!}{([\frac{p}{6}] - 2k)!([\frac{p+1}{3}] + k)!} \pmod{p}.
\end{aligned}$$

If  $p \equiv 1 \pmod{3}$ , using Lemma 2.1(i) we see that

$$\begin{aligned}
\binom{\frac{p-1}{2}}{\frac{p-1}{3} - k} \binom{\frac{p-1}{6} + k}{3k} &= \binom{\frac{p-1}{2}}{\frac{p-1}{6} + k} \binom{\frac{p-1}{6} + k}{3k} = \binom{\frac{p-1}{2}}{3k} \binom{\frac{p-1}{2} - 3k}{\frac{p-1}{6} - 2k} \\
&\equiv \frac{1}{(-4)^{3k}} \binom{6k}{3k} \frac{(\frac{p-1}{2} - 3k)!}{(\frac{p-1}{6} - 2k)! (\frac{p-1}{3} - k)!} \pmod{p}.
\end{aligned}$$

Thus, from the above and Lemma 2.1 we deduce that

$$\begin{aligned}
\frac{\binom{[\frac{p}{6}]}{k} \binom{2[\frac{p}{6}] - 2k}{[\frac{p}{6}]}}{\binom{\frac{p-1}{2}}{\frac{p-1}{3} - k} \binom{\frac{p-1}{6} + k}{3k}} &\equiv (-4)^{[\frac{p}{6}] - 2k + 3k} \frac{\binom{2k}{k} (\frac{p-1}{2} - k)! (\frac{p-1}{3} - k)!}{\binom{6k}{3k} (\frac{p-1}{2} - 3k)! (\frac{p-1}{3} + k)!} \\
&= (-4)^{[\frac{p}{6}] + k} \frac{\binom{2k}{k} \binom{\frac{p-1}{2} - k}{2k}}{\binom{6k}{3k} \binom{\frac{p-1}{3} + k}{2k}} \equiv (-4)^{[\frac{p}{6}] + k} \frac{\binom{2k}{k} \binom{6k}{3k} \binom{3k}{k} / (4^{2k} \binom{2k}{k})}{\binom{6k}{3k} \binom{3k}{k} / ((-27)^k)} \\
&= (-27)^k (-4)^{[\frac{p}{6}] - k} = (-1)^{[\frac{p}{6}]} 3^{3k} 4^{[\frac{p}{6}] - k} \pmod{p}.
\end{aligned}$$

If  $p \equiv 2 \pmod{3}$ , using Lemma 2.1(i) we see that

$$\begin{aligned}
\binom{\frac{p-1}{2}}{\frac{p-2}{3} - k} \binom{\frac{p+1}{6} + k}{3k + 1} &= \binom{\frac{p-1}{2}}{\frac{p+1}{6} + k} \binom{\frac{p+1}{6} + k}{3k + 1} = \binom{\frac{p-1}{2}}{3k + 1} \binom{\frac{p-3}{2} - 3k}{\frac{p-5}{6} - 2k} \\
&\equiv \frac{1}{(-4)^{3k+1}} \binom{6k+2}{3k+1} \frac{(\frac{p-3}{2} - 3k)!}{(\frac{p-5}{6} - 2k)! (\frac{p-2}{3} - k)!} \\
&\equiv \frac{1}{(-4)^{3k} (3k+1)} \binom{6k}{3k} \frac{(\frac{p-1}{2} - 3k)!}{(\frac{p-5}{6} - 2k)! (\frac{p-2}{3} - k)!} \pmod{p}.
\end{aligned}$$

Thus, from the above and Lemma 2.1 we deduce that

$$\begin{aligned}
\frac{\binom{[\frac{p}{6}]}{k} \binom{2[\frac{p}{6}] - 2k}{[\frac{p}{6}]}}{\binom{\frac{p-1}{2}}{\frac{p-2}{3} - k} \binom{\frac{p+1}{6} + k}{3k+1}} &\equiv 3(-4)^{[\frac{p}{6}] - 2k + 3k} \frac{\binom{2k}{k} (\frac{p-1}{2} - k)! (\frac{p-2}{3} - k)!}{\binom{6k}{3k} (\frac{p-1}{2} - 3k)! (\frac{p-2}{3} + k)!} \\
&= 3(-4)^{[\frac{p}{6}] + k} \frac{\binom{2k}{k} \binom{\frac{p-1}{2} - k}{2k}}{\binom{6k}{3k} \binom{\frac{p-2}{3} + k}{2k}} \equiv 3(-4)^{[\frac{p}{6}] + k} \frac{\binom{2k}{k} \binom{6k}{3k} \binom{3k}{k} / (4^{2k} \binom{2k}{k})}{\binom{6k}{3k} \binom{3k}{k} / ((-27)^k)} \\
&= 3(-27)^k (-4)^{[\frac{p}{6}] - k} = (-1)^{[\frac{p}{6}]} 3^{3k+1} 4^{[\frac{p}{6}] - k} \pmod{p}.
\end{aligned}$$

This completes the proof.

**Theorem 2.1.** Let  $p > 3$  be a prime and  $m, n \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then

$$\sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \equiv \begin{cases} -(-3m)^{\frac{p-1}{4}} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. For any positive integer  $k$  it is well known that (see [IR, Lemma 2, p.235])

$$\sum_{x=0}^{p-1} x^k \equiv \begin{cases} p-1 \pmod{p} & \text{if } p-1 \mid k, \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

For  $k, r \in \mathbb{Z}$  with  $0 \leq r \leq k \leq \frac{p-1}{2}$  we have  $0 \leq k+2r \leq \frac{3(p-1)}{2}$ . Thus,

$$\sum_{x=0}^{p-1} x^{k+2r} \equiv \begin{cases} p-1 \pmod{p} & \text{if } k = p-1-2r, \\ 0 \pmod{p} & \text{if } k \neq p-1-2r \end{cases}$$

and therefore

(2.1)

$$\begin{aligned} & \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \\ &= \sum_{x=0}^{p-1} \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} (x^3 + mx)^k n^{\frac{p-1}{2}-k} \\ &= \sum_{x=0}^{p-1} \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \sum_{r=0}^k \binom{k}{r} x^{3r} (mx)^{k-r} n^{\frac{p-1}{2}-k} \\ &= \sum_{r=0}^{(p-1)/2} \sum_{k=r}^{(p-1)/2} \binom{(p-1)/2}{k} \binom{k}{r} m^{k-r} n^{\frac{p-1}{2}-k} \sum_{x=0}^{p-1} x^{k+2r} \\ &\equiv (p-1) \sum_{r=0}^{(p-1)/2} \binom{(p-1)/2}{p-1-2r} \binom{p-1-2r}{r} m^{p-1-3r} n^{2r-\frac{p-1}{2}} \\ &= (p-1) \sum_{\frac{p-1}{4} \leq r \leq \frac{p-1}{3}} \binom{(p-1)/2}{p-1-2r} \binom{p-1-2r}{r} m^{p-1-3r} n^{2r-\frac{p-1}{2}} \pmod{p}. \end{aligned}$$

If  $n \equiv 0 \pmod{p}$ , from the above we deduce that

$$\sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \equiv \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \equiv \begin{cases} -\left(\frac{p-1}{4}\right) m^{\frac{p-1}{4}} \pmod{p} & \text{if } 4 \mid p-1, \\ 0 \pmod{p} & \text{if } 4 \nmid p-1. \end{cases}$$

Thus applying (1.2) and Lemma 2.2 (with  $k = [\frac{p}{12}]$ ) we get

$$P_{[\frac{p}{6}]}(0) = \begin{cases} \frac{1}{(-4)^{[\frac{p}{12}]}} \left( \frac{[\frac{p}{6}]}{[\frac{p}{12}]} \right) \equiv (-1)^{[\frac{p}{12}]} 3^{\frac{p-1}{4}} \left( \frac{\frac{p-1}{2}}{4} \right) \equiv (-3)^{-\frac{p-1}{4}} \left( \frac{\frac{p-1}{2}}{4} \right) \pmod{p} & \text{if } 4 \mid p-1, \\ 0 & \text{if } 4 \nmid p-1. \end{cases}$$

Hence the result is true for  $n \equiv 0 \pmod{p}$ .

Now we assume  $n \not\equiv 0 \pmod{p}$ . From (2.1) we see that

$$\begin{aligned}
\sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) &\equiv \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \\
&\equiv - \left( \frac{n}{p} \right) \sum_{\frac{p-1}{4} \leq r \leq \frac{p-1}{3}} \binom{(p-1)/2}{r} \binom{\frac{p-1}{2} - r}{p-1-3r} \left( \frac{n^2}{m^3} \right)^r \\
&= - \left( \frac{n}{p} \right) \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} \binom{\frac{p-1}{2}}{\lfloor \frac{p}{3} \rfloor - k} \binom{\frac{p-(\frac{p}{3})}{6} + k}{3k + \frac{1-(\frac{p}{3})}{2}} \left( \frac{n^2}{m^3} \right)^{\lfloor \frac{p}{3} \rfloor - k} \pmod{p}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P_{\lfloor \frac{p}{6} \rfloor} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) &= 2^{-\lfloor \frac{p}{6} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} \binom{\lfloor \frac{p}{6} \rfloor}{k} (-1)^k \binom{2\lfloor \frac{p}{6} \rfloor - 2k}{\lfloor \frac{p}{6} \rfloor} \left( \frac{3n\sqrt{-3m}}{2m^2} \right)^{\lfloor \frac{p}{6} \rfloor - 2k} \\
&= 2^{-\lfloor \frac{p}{6} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} \binom{\lfloor \frac{p}{6} \rfloor}{k} (-1)^k \binom{2\lfloor \frac{p}{6} \rfloor - 2k}{\lfloor \frac{p}{6} \rfloor} \left( \frac{3n\sqrt{-3m}}{2m^2} \right)^{\frac{p-(\frac{1}{p})}{2}} \left( -\frac{27n^2}{4m^3} \right)^{\lfloor \frac{p}{12} \rfloor - k} \\
&= (-1)^{\lfloor \frac{p}{12} \rfloor} 2^{-\lfloor \frac{p}{6} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} \binom{\lfloor \frac{p}{6} \rfloor}{k} \binom{2\lfloor \frac{p}{6} \rfloor - 2k}{\lfloor \frac{p}{6} \rfloor} \left( \frac{3n\sqrt{-3m}}{2m^2} \right)^{\frac{p-(\frac{1}{p})}{2}} \left( \frac{27n^2}{4m^3} \right)^{\lfloor \frac{p}{3} \rfloor - k - \frac{p-(\frac{1}{p})}{4}} \\
&\equiv \delta(m, p)^{-1} \left( \frac{n}{p} \right) \left( \frac{3}{p} \right) (-1)^{\lfloor \frac{p}{12} \rfloor} 2^{-\lfloor \frac{p}{6} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} \binom{\lfloor \frac{p}{6} \rfloor}{k} \binom{2\lfloor \frac{p}{6} \rfloor - 2k}{\lfloor \frac{p}{6} \rfloor} \left( \frac{27n^2}{4m^3} \right)^{\lfloor \frac{p}{3} \rfloor - k} \pmod{p},
\end{aligned}$$

where

$$\delta(m, p) = \begin{cases} (-3m)^{\frac{p-1}{4}} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Hence, by the above and Lemma 2.2 we get

$$\begin{aligned}
\delta(m, p) P_{\lfloor \frac{p}{6} \rfloor} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) &\equiv \left( \frac{n}{p} \right) \left( \frac{3}{p} \right) (-1)^{\lfloor \frac{p}{12} \rfloor} 2^{-\lfloor \frac{p}{6} \rfloor} \\
&\times \sum_{k=0}^{\lfloor \frac{p}{12} \rfloor} (-1)^{\lfloor \frac{p}{6} \rfloor} 3^{3k + \frac{1-(\frac{p}{3})}{2}} 4^{\lfloor \frac{p}{6} \rfloor - k} \binom{\frac{p-1}{2}}{\lfloor \frac{p}{3} \rfloor - k} \binom{\frac{p-(\frac{p}{3})}{6} + k}{3k + \frac{1-(\frac{p}{3})}{2}} \left( \frac{27n^2}{4m^3} \right)^{\lfloor \frac{p}{3} \rfloor - k} \pmod{p}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left( \frac{3}{p} \right) (-1)^{\lfloor \frac{p}{12} \rfloor} 2^{-\lfloor \frac{p}{6} \rfloor} (-1)^{\lfloor \frac{p}{6} \rfloor} 3^{3k + \frac{1-(\frac{p}{3})}{2}} 4^{\lfloor \frac{p}{6} \rfloor - k} \left( \frac{27}{4} \right)^{\lfloor \frac{p}{3} \rfloor - k} \\
&= (-1)^{\lfloor \frac{p}{12} \rfloor + \lfloor \frac{p}{6} \rfloor} \left( \frac{3}{p} \right) 2^{2\lfloor \frac{p}{6} \rfloor - 2\lfloor \frac{p}{3} \rfloor} 3^{3\lfloor \frac{p}{3} \rfloor + (1-(\frac{p}{3}))/2} = (-1)^{\lfloor \frac{p}{12} \rfloor + \lfloor \frac{p}{6} \rfloor} \left( \frac{3}{p} \right) 2^{-\frac{p-1}{2}} 3^{p-1} \\
&\equiv (-1)^{\lfloor \frac{p}{12} \rfloor + \lfloor \frac{p}{6} \rfloor} \cdot (-1)^{\frac{p-(\frac{p}{3})}{6}} \cdot (-1)^{-\lfloor \frac{p+1}{4} \rfloor} = (-1)^{2\lfloor \frac{p}{12} \rfloor} = 1 \pmod{p},
\end{aligned}$$

from the above we deduce that

$$\begin{aligned}\delta(m, p) P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) &\equiv \left( \frac{n}{p} \right) \sum_{k=0}^{[\frac{p}{12}]} \left( \frac{\frac{p-1}{2}}{[\frac{p}{3}] - k} \right) \left( \frac{\frac{p-(\frac{p}{3})}{6} + k}{3k + \frac{1-(\frac{p}{3})}{2}} \right) \left( \frac{n^2}{m^3} \right)^{[\frac{p}{3}] - k} \\ &\equiv - \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \pmod{p}.\end{aligned}$$

This completes the proof.

**Remark 2.1** The congruence (2.1) has been given by the author in [S6].

**Corollary 2.1.** *Let  $p \neq 2, 3, 11$  be a prime. Then*

$$P_{[\frac{p}{6}]} \left( \frac{21\sqrt{33}}{121} \right) \equiv \begin{cases} \left( \frac{33}{p} \right) (-33)^{\frac{p-1}{4}} 2a \pmod{p} & \text{if } 4 \mid p-1, p = a^2 + b^2 \text{ and } 4 \mid a-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. By [S5, Corollary 2.1 and (2.2)] we have

$$\begin{aligned}(2.2) \quad &\sum_{x=0}^{p-1} \left( \frac{x^3 - 11x + 14}{p} \right) \\ &= \left( \frac{2}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 4x}{p} \right) = \begin{cases} (-1)^{\frac{p+3}{4}} 2a & \text{if } 4 \mid p-1, p = a^2 + b^2 \text{ and } 4 \mid a-1, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}\end{aligned}$$

Thus, taking  $m = -11$  and  $n = 14$  in Theorem 2.1 we obtain the result.

**Corollary 2.2.** *Let  $p > 5$  be a prime. Then*

$$P_{[\frac{p}{6}]} \left( \frac{7\sqrt{10}}{25} \right) \equiv \begin{cases} (-1)^{\frac{d}{2}} \left( \frac{5}{p} \right) 5^{\frac{p-1}{4}} 2c \pmod{p} & \text{if } 8 \mid p-1, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ \left( \frac{5}{p} \right) 5^{\frac{p-3}{4}} 2d\sqrt{10} \pmod{p} & \text{if } 8 \mid p-3, p = c^2 + 2d^2 \text{ and } 4 \mid d-1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. Using [S6, Lemma 4.2] we see that

$$\begin{aligned}&\sum_{x=0}^{p-1} \left( \frac{x^3 - 30x + 56}{p} \right) \\ &= \sum_{x=0}^{p-1} \left( \frac{(-x)^3 - 30(-x) + 56}{p} \right) = \left( \frac{-1}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 30x - 56}{p} \right) \\ &= \begin{cases} (-1)^{\frac{p+7}{8}} \left( \frac{3}{p} \right) 2c & \text{if } p \equiv 1 \pmod{8}, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ (-1)^{\frac{p-3}{8}} \left( \frac{3}{p} \right) 2c & \text{if } p \equiv 3 \pmod{8}, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ 0 & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}\end{aligned}$$

By [S3, p.1317] we have

$$2^{[\frac{p}{4}]} \equiv \begin{cases} (-1)^{\frac{c^2-1}{8}} = (-1)^{\frac{p-1}{8} + \frac{d}{2}} \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1 \pmod{8}, \\ (-1)^{\frac{c^2-1}{8}} \frac{d}{c} = (-1)^{\frac{p-3}{8}} \frac{d}{c} \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 3 \pmod{8} \text{ with } 4 \mid c-d. \end{cases}$$

Now taking  $m = -30$  and  $n = 56$  in Theorem 2.1 and applying the above we deduce the result.

**Corollary 2.3.** *Let  $p > 5$  be a prime. Then*

$$P_{[\frac{p}{6}]} \left( \frac{11\sqrt{5}}{25} \right) \equiv \begin{cases} 5^{\frac{p-1}{4}} \left( \frac{5}{p} \right) 2A \pmod{p} & \text{if } 12 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ -5^{\frac{p-3}{4}} \left( \frac{5}{p} \right) 2A\sqrt{5} \pmod{p} & \text{if } 12 \mid p-7, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. By [S2, Lemma 2.3] (or [S5, Corollary 2.1 (with  $t = 5/3$ ) and (2.3)]) we have

$$(2.3) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 15x + 22}{p} \right) = \begin{cases} -2A & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Thus, taking  $m = -15$  and  $n = 22$  in Theorem 2.1 we obtain the result.

**Corollary 2.4.** *Let  $p > 5$  be a prime. Then*

$$P_{[\frac{p}{6}]} \left( \frac{253\sqrt{10}}{800} \right) \equiv \begin{cases} -\left( \frac{10}{p} \right) 10^{\frac{p-1}{4}} L \pmod{p} & \text{if } 12 \mid p-1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ \left( \frac{10}{p} \right) 10^{\frac{p-3}{4}} L\sqrt{10} \pmod{p} & \text{if } 12 \mid p-7, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. From [S6, Corollary 3.3] we know that

$$(2.4) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 120x + 506}{p} \right) = \begin{cases} \left( \frac{2}{p} \right) L & \text{if } 3 \mid p-1, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-1, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Thus taking  $m = -120$  and  $n = 506$  in Theorem 2.1 we deduce the result.

**Corollary 2.5.** *Let  $p > 7$  be a prime. Then*

$$P_{[\frac{p}{6}]} \left( \frac{3\sqrt{105}}{25} \right) \equiv \begin{cases} 2\left( \frac{p}{15} \right) 15^{\frac{p-1}{4}} C \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid C-1, \\ 2\left( \frac{p}{15} \right) 15^{\frac{p-3}{4}} D\sqrt{105} \pmod{p} & \text{if } p \equiv 11, 15, 23 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid D-1, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. Since  $(-x-7)^3 - 35(-x-7) + 98 = -(x^3 + 21x^2 + 112x)$ , from [R1,R2] we see that

$$(2.5) \quad \begin{aligned} \sum_{x=0}^{p-1} \left( \frac{x^3 - 35x + 98}{p} \right) &= (-1)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left( \frac{x^3 + 21x^2 + 112x}{p} \right) \\ &= \begin{cases} (-1)^{\frac{p+1}{2}} 2\left( \frac{C}{7} \right) C & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and } p = C^2 + 7D^2, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Suppose  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = C^2 + 7D^2$ . By [S3, p.1317] we have

$$(2.6) \quad 7^{\left[ \frac{p}{4} \right]} \equiv \begin{cases} \left( \frac{C}{7} \right) \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28} \text{ and } C \equiv 1 \pmod{4}, \\ -\left( \frac{C}{7} \right) \frac{D}{C} \pmod{p} & \text{if } p \equiv 11, 15, 23 \pmod{28} \text{ and } D \equiv 1 \pmod{4}. \end{cases}$$

Now taking  $m = -35$  and  $n = 98$  in Theorem 2.1 and applying all the above we deduce the result.



**Corollary 2.6.** *Let  $p$  be a prime such that  $p \neq 2, 3, 5, 7, 17$ .*

- (i) *If  $p \equiv 3, 5, 6 \pmod{7}$ , then  $P_{[\frac{p}{6}]}(\frac{171\sqrt{1785}}{85^2}) \equiv 0 \pmod{p}$ .*
- (ii) *If  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = C^2 + 7D^2$  for some  $C, D \in \mathbb{Z}$ , then*

$$P_{[\frac{p}{6}]}(\frac{171\sqrt{1785}}{85^2}) \equiv \begin{cases} (\frac{255}{p})255^{\frac{p-1}{4}} \cdot 2C \pmod{p} & \text{if } 4 \mid p-1 \text{ and } 4 \mid C-1, \\ (\frac{255}{p})255^{\frac{p-3}{4}} \cdot 2D\sqrt{1785} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } 4 \mid D-1. \end{cases}$$

Proof. From [W, p.296] we know that

$$\begin{aligned} & \sum_{x=0}^{p-1} \left( \frac{(x^2 + 6x + 2)(3x^2 + 16x)}{p} \right) \\ &= \begin{cases} -2(\frac{-2}{p})(\frac{C}{7})C - (\frac{3}{p}) & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and } p = C^2 + 7D^2, \\ -(\frac{3}{p}) & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

As  $(x^2 + 6x + 2)(3x^2 + 16x) = x^4(3 + 34/x + 102/x^2 + 32/x^3)$ , we see that

$$\begin{aligned} & \sum_{x=0}^{p-1} \left( \frac{(x^2 + 6x + 2)(3x^2 + 16x)}{p} \right) \\ &= \sum_{x=1}^{p-1} \left( \frac{3 + 34/x + 102/x^2 + 32/x^3}{p} \right) = \sum_{x=1}^{p-1} \left( \frac{3 + 34x + 102x^2 + 32x^3}{p} \right) \\ &= \left( \frac{2}{p} \right) \sum_{x=1}^{p-1} \left( \frac{6 + 68x + 204x^2 + 64x^3}{p} \right) = \left( \frac{2}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 + \frac{51}{4}x^2 + 17x + 6}{p} \right) - \left( \frac{12}{p} \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{x=0}^{p-1} \left( \frac{x^3 + \frac{51}{4}x^2 + 17x + 6}{p} \right) \\ &= \sum_{x=0}^{p-1} \left( \frac{(x - \frac{17}{4})^3 + \frac{51}{4}(x - \frac{17}{4})^2 + 17(x - \frac{17}{4}) + 6}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{595}{16}x + \frac{5586}{64}}{p} \right) \\ &= \sum_{x=0}^{p-1} \left( \frac{(\frac{x}{4})^3 - \frac{595}{16} \cdot \frac{x}{4} + \frac{5586}{64}}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{x^3 - 595x + 5586}{p} \right). \end{aligned}$$

Now combining all the above we deduce

$$(2.7) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 595x + 5586}{p} \right) = \begin{cases} (-1)^{\frac{p+1}{2}} 2C(\frac{C}{7}) & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Taking  $m = -595$  and  $n = 5586$  in Theorem 2.1 and then applying (2.7) and (2.6) we deduce the result.

**Corollary 2.7.** *Let  $p \neq 2, 3, 11$  be a prime.*

- (i) *If  $p \equiv 2, 6, 7, 8, 10 \pmod{11}$ , then  $P_{[\frac{p}{6}]}(\frac{7}{32}\sqrt{22}) \equiv 0 \pmod{p}$ .*
- (ii) *If  $p \equiv 1, 3, 4, 5, 9 \pmod{11}$  and hence  $4p = u^2 + 11v^2$  for some  $u, v \in \mathbb{Z}$ , then*

$$P_{[\frac{p}{6}]}(\frac{7}{32}\sqrt{22}) \equiv \begin{cases} -(\frac{p}{3})(-2)^{\frac{p-1}{4}}u \pmod{p} & \text{if } 4 \mid p-1 \text{ and } 4 \mid u-1, \\ (\frac{p}{3})2^{\frac{p-1}{4}}u \pmod{p} & \text{if } 4 \mid p-1 \text{ and } 8 \mid u-2, \\ -(\frac{p}{3})(-2)^{\frac{p-3}{4}}v\sqrt{22} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } 4 \mid v-1, \\ (\frac{p}{3})2^{\frac{p-3}{4}}v\sqrt{22} \pmod{p} & \text{if } 4 \mid p-3 \text{ and } 8 \mid v-2. \end{cases}$$

Proof. It is known that (see [RP] and [JM])

$$(2.8) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 96 \cdot 11x + 112 \cdot 11^2}{p} \right) = \begin{cases} (\frac{2}{p})(\frac{u}{11})u & \text{if } (\frac{p}{11}) = 1 \text{ and } 4p = u^2 + 11v^2, \\ 0 & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

Thus applying Theorem 2.1 we deduce

$$\begin{aligned} & P_{[\frac{p}{6}]}(\frac{7}{32}\sqrt{22}) \\ & \equiv \begin{cases} -(\frac{p}{3})22^{\frac{p-1}{4}}(\frac{u}{11})u \pmod{p} & \text{if } (\frac{p}{11})=1, 4 \mid p-1 \text{ and } 4p = u^2 + 11v^2, \\ -(\frac{p}{3})22^{\frac{p-3}{4}}(\frac{u}{11})u\sqrt{22} \pmod{p} & \text{if } (\frac{p}{11})=1, 4 \mid p-3 \text{ and } 4p = u^2 + 11v^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{11}) = -1. \end{cases} \end{aligned}$$

Now assume  $(\frac{p}{11}) = 1$  and so  $4p = u^2 + 11v^2$ . If  $u \equiv v \equiv 1 \pmod{4}$ , by [S3, Theorem 4.3] we have

$$(-11)^{[\frac{p}{4}]} \equiv \begin{cases} (\frac{u}{11}) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ (\frac{u}{11})\frac{v}{u} \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

If  $u \equiv v \equiv 0 \pmod{2}$ , by [S3, Corollary 4.6] we have

$$11^{[\frac{p}{4}]} \equiv \begin{cases} -(\frac{u}{11}) \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and } 8 \mid u-2, \\ -(\frac{u}{11})\frac{v}{u} \pmod{p} & \text{if } p \equiv 3 \pmod{4} \text{ and } 8 \mid v-2. \end{cases}$$

Now combining all the above we derive the result.

From [RPR], [JM] and [PV] we know that for any prime  $p > 3$ ,

$$\begin{aligned} (2.9) \quad & \sum_{x=0}^{p-1} \left( \frac{x^3 - 8 \cdot 19x + 2 \cdot 19^2}{p} \right) = \begin{cases} (\frac{2}{p})(\frac{u}{19})u & \text{if } (\frac{p}{19}) = 1 \text{ and } 4p = u^2 + 19v^2, \\ 0 & \text{if } (\frac{p}{19}) = -1, \end{cases} \\ & \sum_{x=0}^{p-1} \left( \frac{x^3 - 80 \cdot 43x + 42 \cdot 43^2}{p} \right) = \begin{cases} (\frac{2}{p})(\frac{u}{43})u & \text{if } (\frac{p}{43}) = 1 \text{ and } 4p = u^2 + 43v^2, \\ 0 & \text{if } (\frac{p}{43}) = -1, \end{cases} \\ & \sum_{x=0}^{p-1} \left( \frac{x^3 - 440 \cdot 67x + 434 \cdot 67^2}{p} \right) = \begin{cases} (\frac{2}{p})(\frac{u}{67})u & \text{if } (\frac{p}{67}) = 1 \text{ and } 4p = u^2 + 67v^2, \\ 0 & \text{if } (\frac{p}{67}) = -1, \end{cases} \\ & \sum_{x=0}^{p-1} \left( \frac{x^3 - 80 \cdot 23 \cdot 29 \cdot 163x + 14 \cdot 11 \cdot 19 \cdot 127 \cdot 163^2}{p} \right) \\ & = \begin{cases} (\frac{2}{p})(\frac{u}{163})u & \text{if } (\frac{p}{163}) = 1 \text{ and } 4p = u^2 + 163v^2, \\ 0 & \text{if } (\frac{p}{163}) = -1. \end{cases} \end{aligned}$$

Thus, using the method in the proof of Corollary 2.7 one can similarly determine  $P_{[\frac{p}{6}]}(\frac{3}{32}\sqrt{114})$ ,  $P_{[\frac{p}{6}]}(\frac{63\sqrt{645}}{1600})$ ,  $P_{[\frac{p}{6}]}(\frac{651}{96800}\sqrt{22110})$ ,  $P_{[\frac{p}{6}]}(\frac{557403}{26680^2}\sqrt{1630815}) \pmod{p}$ .

**Lemma 2.3.** *Let  $p$  be a prime greater than 3, and let  $x$  be a variable. Then*

$$P_{[\frac{p}{6}]}(x) \equiv \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left(\frac{1-x}{864}\right)^k \pmod{p}.$$

Proof. Suppose that  $r \in \{1, 5\}$  is given by  $p \equiv r \pmod{6}$ . Then clearly

$$\begin{aligned} \binom{[\frac{p}{6}] + k}{2k} &= \frac{(\frac{p-r}{6} + k)(\frac{p-r}{6} + k - 1) \cdots (\frac{p-r}{6} - k + 1)}{(2k)!} \\ &= \frac{(p + 6k - r)(p + 6k - r - 6) \cdots (p - (6k + r - 6))}{6^{2k} \cdot (2k)!} \\ &\equiv (-1)^k \frac{(6k - r)(6k - r - 6) \cdots (6 - r) \cdot r(r + 6) \cdots (6k + r - 6)}{6^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (6k)!}{(2 \cdot 4 \cdots 6k)(3 \cdot 9 \cdot 15 \cdots (6k - 3)) \cdot 6^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (6k)!}{2^{3k}(3k)! \cdot 3^k \frac{(2k)!}{2 \cdot 4 \cdot 6 \cdots 2k} \cdot 36^k (2k)!} \equiv \frac{(6k)!k!}{(-432)^k (3k)!(2k)!^2} \pmod{p}. \end{aligned}$$

Hence

$$(2.10) \quad \binom{[\frac{p}{6}]}{k} \binom{[\frac{p}{6}] + k}{k} = \binom{[\frac{p}{6}] + k}{2k} \binom{2k}{k} \equiv \frac{\binom{6k}{3k} \binom{3k}{k}}{(-432)^k} \pmod{p}.$$

This together with (1.3) yields the result.

**Theorem 2.2.** *Let  $p > 3$  be a prime and  $m, n \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$P_{[\frac{p}{6}]}(\frac{n}{2m^3}) \equiv \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left(\frac{2m^3 - n}{12^3 m^3}\right)^k \equiv -\left(\frac{3m}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3m^2 x + n}{p}\right) \pmod{p}.$$

Proof. Replacing  $m$  by  $-3m^2$  in Theorem 2.1 and then applying Lemma 2.3 we deduce the result.

For positive integers  $a_1, a_2, a_3, a_4$  let

$$q \prod_{k=1}^{\infty} (1 - q^{a_1 k})(1 - q^{a_2 k})(1 - q^{a_3 k})(1 - q^{a_4 k}) = \sum_{n=1}^{\infty} c(a_1, a_2, a_3, a_4; n) q^n \quad (|q| < 1).$$

For  $(a_1, a_2, a_3, a_4) = (1, 1, 11, 11), (2, 2, 10, 10), (1, 3, 5, 15), (1, 2, 7, 14)$  and  $(4, 4, 8, 8)$  it is known that (see [MO, Theorem 1])

$$f(z) = \sum_{n=1}^{\infty} c(a_1, a_2, a_3, a_4; n) q^n \quad (q = e^{2\pi i z})$$

are weight 2 newforms.

**Corollary 2.8.** *Let  $p$  be an odd prime. Then*

$$c(1, 1, 11, 11; p) \equiv P_{[\frac{p}{6}]} \left( \frac{19}{8} \right) \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{[p/6]} \frac{\binom{6k}{3k} \binom{3k}{k}}{256^k} \pmod{p}.$$

Proof. It is easy to see that the result holds for  $p = 3, 11$ . Now assume  $p \neq 3, 11$ . By the well known work of Eichler in 1954, we have

$$|\{(x, y) \in F_p \times F_p : y^2 + y = x^3 - x^2\}| = p - c(1, 1, 11, 11; p).$$

Since

$$\begin{aligned} & |\{(x, y) \in F_p \times F_p : y^2 + y = x^3 - x^2\}| \\ &= \left| \left\{ (x, y) \in F_p \times F_p : \left(y + \frac{1}{2}\right)^2 = x^3 - x^2 + \frac{1}{4} \right\} \right| \\ &= \left| \left\{ (x, y) \in F_p \times F_p : y^2 = x^3 - x^2 + \frac{1}{4} \right\} \right| \\ &= p + \sum_{x=0}^{p-1} \left( \frac{x^3 - x^2 + \frac{1}{4}}{p} \right) = p + \sum_{x=0}^{p-1} \left( \frac{(x + \frac{1}{3})^3 - (x + \frac{1}{3})^2 + \frac{1}{4}}{p} \right) \\ &= p + \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{1}{3}x + \frac{19}{108}}{p} \right) = p + \sum_{x=0}^{p-1} \left( \frac{(\frac{x}{6})^3 - \frac{1}{3} \cdot \frac{x}{6} + \frac{19}{108}}{p} \right) \\ &= p + \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 12x + 38}{p} \right), \end{aligned}$$

we obtain

$$(2.11) \quad c(1, 1, 11, 11; p) = - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 12x + 38}{p} \right).$$

Using Theorem 2.2 we see that

$$c(1, 1, 11, 11; p) = - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 12x + 38}{p} \right) \equiv P_{[\frac{p}{6}]} \left( \frac{19}{8} \right) \pmod{p}.$$

From (1.2) and Lemma 2.3 we have

$$\begin{aligned} P_{[\frac{p}{6}]} \left( \frac{19}{8} \right) &= (-1)^{[\frac{p}{6}]} P_{[\frac{p}{6}]} \left( -\frac{19}{8} \right) \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left( \frac{1 + 19/8}{864} \right)^k \\ &= (-1)^{\frac{p-1}{2}} \sum_{k=0}^{[p/6]} \frac{\binom{6k}{3k} \binom{3k}{k}}{256^k} \pmod{p}. \end{aligned}$$

Thus the result follows.

**Conjecture 2.1.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned}
c(2, 2, 10, 10; p) &= -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 12x - 11}{p}\right), \\
c(2, 4, 6, 12; p) &= -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 39x - 70}{p}\right), \\
c(1, 3, 5, 15; p) &= -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3x - 322}{p}\right), \\
c(1, 2, 7, 14; p) &= -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 75x - 506}{p}\right), \\
c(4, 4, 8, 8; p) &= -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 99x - 378}{p}\right).
\end{aligned}$$

If  $p > 3$  is a prime of the form  $4k + 3$ , from Conjecture 2.1 and [S2, Theorem 2.8] we deduce that

$$\begin{aligned}
& p + 1 - \left(\frac{p}{3}\right) c(2, 2, 10, 10; p) \\
&= p + 1 + \sum_{x=0}^{p-1} \left(\frac{x^3 - 12x - 11}{p}\right) = \#E_p(x^3 - 12x - 11) \\
&= \begin{cases} 4N_p - \frac{3p-1}{2} - 2\delta(p) & \text{if } p \equiv 7 \pmod{12}, \\ -4N_p + \frac{7p+3}{2} + 2\delta(p) & \text{if } p \equiv 11 \pmod{12}, \end{cases}
\end{aligned}$$

where  $N_p$  is the number of  $a \in \{0, 1, \dots, p-1\}$  such that  $x^4 - 4x^2 + 4x \equiv a \pmod{p}$  is solvable, and

$$\delta(p) = \begin{cases} 0 & \text{if } p \equiv 7, 23 \pmod{40}, \\ 1 & \text{if } p \equiv 3, 27, 31, 39 \pmod{40}, \\ 2 & \text{if } p \equiv 11, 19 \pmod{40}. \end{cases}$$

Hence

$$(2.12) \quad c(2, 2, 10, 10; p) = \frac{5p+1}{2} - 4N_p + 2\delta(p) \quad \text{for } p \equiv 3 \pmod{4}.$$

**Theorem 2.3.** *Let  $p > 3$  be a prime, and let  $t$  be a variable. Then*

$$(2.13) \quad P_{[\frac{p}{6}]}(t) \equiv -\left(\frac{3}{p}\right) \sum_{x=0}^{p-1} (x^3 - 3x + 2t)^{\frac{p-1}{2}} \pmod{p}.$$

Proof. Taking  $m = 1$  and  $n = 2t$  in Theorem 2.2 we see that (2.13) is true for  $t = 0, 1, \dots, p-1$ . Since both sides of (2.13) are polynomials in  $t$  with degree less than  $(p-1)/2$ , applying Lagrange's theorem we see that (2.13) holds when  $t$  is a variable.

**Theorem 2.4.** *Let  $p > 3$  be a prime and let  $t$  be a variable.*

(i) *If  $t^2 + 3 \not\equiv 0 \pmod{p}$ , then*

$$P_{\frac{p-1}{2}}(t) \equiv \begin{cases} (-t^2 - 3)^{\frac{p-1}{4}} P_{[\frac{p}{6}]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2}) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{(-t^2-3)^{\frac{p+1}{4}}}{\sqrt{t^2+3}} P_{[\frac{p}{6}]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2}) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(ii) *If  $3t + 5 \not\equiv 0 \pmod{p}$ , then*

$$P_{[\frac{p}{4}]}(t) \equiv \begin{cases} (6t + 10)^{\frac{p-1}{4}} P_{[\frac{p}{6}]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2}) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{(6t+10)^{\frac{p+1}{4}}}{\sqrt{6t+10}} P_{[\frac{p}{6}]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2}) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. Since both sides are polynomials of  $t$  with degree at most  $p-3$ . It suffices to show that the congruences are true for  $t \in R_p$ . Now combining (1.4)-(1.5) with Theorem 2.1 we deduce the result.

**Corollary 2.9.** *Let  $p > 3$  be a prime and  $m \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\begin{aligned} P_{\frac{p-1}{2}}(m^2 - 3) &\equiv \left(\frac{-2m}{p}\right) P_{[\frac{p}{6}]}(\frac{(m^2-3)(m^2-6)}{m}) \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \left(\frac{4-m^2}{32}\right)^k \\ &\equiv \left(\frac{2m}{p}\right) \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left(\frac{m^4 - 9m^2 - m + 18}{864m}\right)^k \pmod{p} \end{aligned}$$

and

$$\begin{aligned} P_{[\frac{p}{4}]}(\frac{2m^2-5}{3}) &\equiv \left(\frac{2m}{p}\right) P_{[\frac{p}{6}]}(\frac{3m^2-4}{m^3}) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{4-m^2}{192}\right)^k \\ &\equiv \left(\frac{2m}{p}\right) \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left(\frac{(m+1)(m-2)^2}{864m^3}\right)^k \pmod{p}. \end{aligned}$$

Proof. Taking  $t = m^2 - 3$  in Theorem 2.4(i) and then applying [S4, (2.4)] and Lemma 2.3 we deduce the first congruence. Taking  $t = (2m^2 - 5)/3$  in Theorem 2.4(ii) and then applying [S5, Theorem 2.1(ii)] and Lemma 2.3 we deduce the second congruence.

**Theorem 2.5.** *Let  $p > 3$  be a prime and let  $t$  be a variable. Then*

$$P_{[\frac{p}{3}]}(t) \equiv \begin{cases} (5-4t)^{\frac{p-1}{4}} P_{[\frac{p}{6}]}(\frac{2t^2-14t+11}{(5-4t)^2} \sqrt{5-4t}) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{(5-4t)^{\frac{p+1}{4}}}{\sqrt{5-4t}} P_{[\frac{p}{6}]}(\frac{2t^2-14t+11}{(5-4t)^2} \sqrt{5-4t}) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. Since both sides are polynomials in  $t$  with degree at most  $p-2$ . It suffices to show that the congruence is true for all  $t \in R_p$  with  $t \not\equiv \frac{5}{4} \pmod{p}$ . Set  $m = 3(4t-5)$  and  $n = 2(2t^2 - 14t + 11)$ . Then

$$\frac{3n\sqrt{-3m}}{2m^2} = \frac{(2t^2 - 14t + 11)\sqrt{5-4t}}{(5-4t)^2}.$$

Thus, by (1.6) and Theorem 2.1 we have

$$P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\ \equiv \begin{cases} \left(\frac{p}{3}\right)(9(5t-4))^{\frac{p-1}{4}} P_{[\frac{p}{6}]} \left(\frac{2t^2-14t+11}{(5-4t)^2} \sqrt{5-4t}\right) \pmod{p} & \text{if } 4 \mid p-1, \\ \left(\frac{p}{3}\right) \frac{(9(5t-4))^{\frac{p+1}{4}}}{\sqrt{9(5-4t)}} P_{[\frac{p}{6}]} \left(\frac{2t^2-14t+11}{(5-4t)^2} \sqrt{5-4t}\right) \pmod{p} & \text{if } 4 \mid p-3. \end{cases}$$

For  $p \equiv 1 \pmod{4}$  we have  $9^{\frac{p-1}{4}} \left(\frac{p}{3}\right) \equiv \left(\frac{3}{p}\right) \left(\frac{p}{3}\right) = 1 \pmod{p}$ , For  $p \equiv 3 \pmod{4}$  we have  $9^{\frac{p+1}{4}} \frac{1}{3} \left(\frac{p}{3}\right) \equiv \left(\frac{3}{p}\right) \left(\frac{p}{3}\right) = -1 \pmod{p}$ . Thus the result follows.

**Corollary 2.10.** *Let  $p > 3$  be a prime and  $m \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$P_{[\frac{p}{3}]} \left(\frac{5-m^2}{4}\right) \equiv \left(\frac{m}{p}\right) P_{[\frac{p}{6}]} \left(\frac{m^4 + 18m^2 - 27}{8m^3}\right) \equiv \sum_{k=0}^{[p/3]} \binom{2k}{k} \binom{3k}{k} \left(\frac{m^2-1}{216}\right)^k \\ \equiv \left(\frac{-m}{p}\right) \sum_{k=0}^{[p/6]} \binom{6k}{3k} \binom{3k}{k} \left(\frac{(m+1)(3-m)^3}{2^8 \cdot 3^3 m^3}\right)^k \pmod{p}.$$

Proof. Taking  $t = \frac{5-m^2}{4}$  in Theorem 2.5 and then applying [S4, Lemma 2.3] and Lemma 2.3 we deduce the result.

**Corollary 2.11.** *Let  $p > 3$  be a prime. Then*

$$P_{[\frac{p}{3}]} \left(\frac{7 \pm 3\sqrt{3}}{2}\right) \\ \equiv \begin{cases} 2a(-3 \pm 2\sqrt{3})^{\frac{p-1}{4}} \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = a^2 + b^2 \text{ with } a \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. Set  $t = (7 \pm 3\sqrt{3})/2$ . Then  $2t^2 - 14t + 11 = 0$ . Thus, from Theorem 2.5 and the congruence for  $P_{[\frac{p}{6}]}(0)$  in the proof of Theorem 2.1 we deduce

$$P_{[\frac{p}{3}]} \left(\frac{7 \pm 3\sqrt{3}}{2}\right) \equiv \begin{cases} (-9 \mp 6\sqrt{3})^{\frac{p-1}{4}} \equiv (-3 \pm 2\sqrt{3})^{\frac{p-1}{4}} \left(\frac{p-1}{\frac{p-1}{4}}\right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

It is well known that  $\left(\frac{p-1}{\frac{p-1}{4}}\right) \equiv 2a \pmod{p}$  for  $p \equiv 1 \pmod{4}$  (see [BEW, p.269]). Thus the corollary is proved.

**Theorem 2.6.** *Let  $p > 3$  be a prime and  $m, n \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p}\right) \\ \equiv \begin{cases} -(-3m)^{\frac{p-1}{4}} \sum_{k=0}^{[p/12]} \binom{[p/12]}{k} \binom{[5p/12]}{k} \left(\frac{4m^3+27n^2}{4m^3}\right)^k \pmod{p} & \text{if } 4 \mid p-1, \\ -\frac{3n}{2m^2} (-3m)^{\frac{p+1}{4}} \sum_{k=0}^{[p/12]} \binom{[p/12]}{k} \binom{[5p/12]}{k} \left(\frac{4m^3+27n^2}{4m^3}\right)^k \pmod{p} & \text{if } 4 \mid p-3. \end{cases}$$

Proof. Let  $P_n^{(\alpha, \beta)}(x)$  be the Jacobi polynomial defined by

$$P_n^{(\alpha, \beta)}(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x+1)^k (x-1)^{n-k}.$$

It is known that (see [AAR, p.315])

$$(2.14) \quad P_{2n}(x) = P_n^{(0, -\frac{1}{2})}(2x^2 - 1) \quad \text{and} \quad P_{2n+1}(x) = xP_n^{(0, \frac{1}{2})}(2x^2 - 1).$$

From [B, p.170] we know that

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \binom{n+\alpha}{n} \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k \cdot k!} \left(\frac{1-x}{2}\right)^k \\ &= \binom{n+\alpha}{n} \sum_{k=0}^n \frac{\binom{n}{k} \binom{-n-\alpha-\beta-1}{k}}{\binom{-1-\alpha}{k}} \left(\frac{x-1}{2}\right)^k. \end{aligned}$$

Thus,

$$(2.15) \quad P_n^{(0, \beta)}(x) = \sum_{k=0}^n \binom{n}{k} \binom{-n-\beta-1}{k} \left(\frac{1-x}{2}\right)^k.$$

Hence, if  $p \equiv 1 \pmod{4}$ , then  $[\frac{p}{6}] = 2[\frac{p}{12}]$  and so

$$\begin{aligned} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) &= P_{[\frac{p}{12}]}^{(0, -\frac{1}{2})} \left( 2 \cdot \frac{-27n^2}{4m^3} - 1 \right) = \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{-\frac{1}{2}}{k} \left( 1 - \frac{-27n^2}{4m^3} \right)^k \\ &\equiv \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{\frac{p-1}{2}}{k} \left( \frac{4m^3 + 27n^2}{4m^3} \right)^k \\ &= \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{[\frac{5p}{12}]}{k} \left( \frac{4m^3 + 27n^2}{4m^3} \right)^k \pmod{p}; \end{aligned}$$

if  $p \equiv 3 \pmod{4}$ , then  $[\frac{p}{6}] = 2[\frac{p}{12}] + 1$  and so

$$\begin{aligned} P_{[\frac{p}{6}]} \left( \frac{3n\sqrt{-3m}}{2m^2} \right) &= \frac{3n\sqrt{-3m}}{2m^2} P_{2[\frac{p}{12}]+1}^{(0, \frac{1}{2})} \left( 2 \cdot \frac{-27n^2}{4m^3} - 1 \right) \\ &= \frac{3n\sqrt{-3m}}{2m^2} \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{-\frac{3}{2}}{k} \left( 1 - \frac{-27n^2}{4m^3} \right)^k \\ &\equiv \frac{3n\sqrt{-3m}}{2m^2} \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{\frac{p-3}{2}}{k} \left( \frac{4m^3 + 27n^2}{4m^3} \right)^k \\ &= \frac{3n\sqrt{-3m}}{2m^2} \sum_{k=0}^{[\frac{p}{12}]} \binom{[\frac{p}{12}]}{k} \binom{[\frac{5p}{12}]}{k} \left( \frac{4m^3 + 27n^2}{4m^3} \right)^k \pmod{p}. \end{aligned}$$

Now combining the above with Theorem 2.1 we deduce the result.



### 3. A general congruence modulo $p^2$ .

**Lemma 3.1.** *For any nonnegative integer  $n$  we have*

$$\sum_{k=0}^n \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{n-k} (-432)^{n-k} = \sum_{k=0}^n \binom{3k}{k} \binom{6k}{3k} \binom{3(n-k)}{n-k} \binom{6(n-k)}{3(n-k)}.$$

Proof. Let  $m$  be a nonnegative integer. For  $k \in \{0, 1, \dots, m\}$  set

$$F_1(m, k) = \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k},$$

$$F_2(m, k) = \binom{3k}{k} \binom{6k}{3k} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)}.$$

For  $k \in \{0, 1, \dots, m+1\}$  set

$$G_1(m, k) = \frac{186624k^2(m+2)(m+1-2k)}{k+1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k+1}{m+2-k} (-432)^{m-k},$$

$$G_2(m, k) = \frac{12k^2(36m^2 - 36mk + 129m - 62k + 114)}{(m-k+2)^2}$$

$$\times \binom{3k}{k} \binom{6k}{3k} \binom{3(m-k+1)}{m-k+1} \binom{6(m-k+1)}{3(m-k+1)}.$$

For  $i = 1, 2$  and  $k \in \{0, 1, \dots, m\}$ , it is easy to check that

$$(3.1) \quad (m+2)^3 F_i(m+2, k) - 24(2m+3)(18m^2 + 54m + 41) F_i(m+1, k) \\ + 20736(m+1)(3m+1)(3m+5) F_i(m, k) = G_i(m, k+1) - G_i(m, k).$$

Set  $S_i(n) = \sum_{k=0}^n F_i(n, k)$  for  $n = 0, 1, 2, \dots$ . Then

$$(m+2)^3 (S_i(m+2) - F_i(m+2, m+2) - F_i(m+2, m+1)) \\ - 24(2m+3)(18m^2 + 54m + 41) (S_i(m+1) - F_i(m+1, m+1)) \\ + 20736(m+1)(3m+1)(3m+5) S_i(m) \\ = (m+2)^3 \sum_{k=0}^m F_i(m+2, k) - 24(2m+3)(18m^2 + 54m + 41) \sum_{k=0}^m F_i(m+1, k) \\ + 20736(m+1)(3m+1)(3m+5) \sum_{k=0}^m F_i(m, k) \\ = \sum_{k=0}^m (G_i(m, k+1) - G_i(m, k)) = G_i(m, m+1) - G_i(m, 0) = G_i(m, m+1).$$

Thus, for  $i = 1, 2$  and  $m = 0, 1, 2, \dots$ ,

$$(3.2) \quad (m+2)^3 S_i(m+2) - 24(2m+3)(18m^2 + 54m + 41) S_i(m+1) \\ + 20736(m+1)(3m+1)(3m+5) S_i(m) \\ = G_i(m, m+1) + (m+2)^3 (F_i(m+2, m+2) + F_i(m+2, m+1)) \\ - 24(2m+3)(18m^2 + 54m + 41) F_i(m+1, m+1) = 0.$$

Since  $S_1(0) = 1 = S_2(0)$  and  $S_1(1) = 120 = S_2(1)$ , from (3.2) we deduce  $S_1(n) = S_2(n)$  for all  $n = 0, 1, 2, \dots$ . This completes the proof.

For given prime  $p$  and integer  $n$ , if  $p^\alpha \mid n$  but  $p^{\alpha+1} \nmid n$ , we say that  $p^\alpha \parallel n$ .

**Lemma 3.2.** *Let  $p$  be an odd prime and  $k, r \in \{0, 1, \dots, p-1\}$  with  $k+r \geq p$ . Then*

$$\binom{3k}{k} \binom{6k}{3k} \binom{3r}{r} \binom{6r}{3r} \equiv 0 \pmod{p^2}.$$

Proof. If  $k > \frac{5p}{6}$ , then  $p^5 \mid (6k)!$ ,  $p \parallel (2k)!$ ,  $p^2 \parallel (3k)!$  and so  $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p^2}$ . If  $\frac{2p}{3} \leq k < \frac{5p}{6}$ , then  $2p \leq 3k < 3p$ ,  $4p \leq 6k < 5p$ ,  $p^4 \parallel (6k)!$ ,  $p^2 \parallel (3k)!$ ,  $p \parallel (2k)!$  and so  $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$ . If  $\frac{p}{2} < k < \frac{2p}{3}$ , then  $p < 3k < 2p$ ,  $3p < 6k < 4p$ ,  $p^3 \mid (6k)!$ ,  $p \parallel (2k)!$ ,  $p \parallel (3k)!$  and so  $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$ . If  $\frac{p}{3} \leq k < \frac{p}{2}$ , then  $2k < p \leq 3k < 2p \leq 6k$ ,  $p^2 \mid (6k)!$ ,  $p \nmid (2k)!$ ,  $p \parallel (3k)!$  and so  $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$ . If  $\frac{p}{6} < k < \frac{p}{3}$ , then  $3k < p < 6k$  and so  $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$ .

From the above we see that  $p \mid \binom{3k}{k} \binom{6k}{3k}$  for  $k > \frac{p}{6}$ . Therefore, if  $k > \frac{p}{6}$  and  $r > \frac{p}{6}$ , then  $\binom{3k}{k} \binom{6k}{3k} \binom{3r}{r} \binom{6r}{3r} \equiv 0 \pmod{p^2}$ . If  $r < \frac{p}{6}$ , then  $k \geq p-r > \frac{5p}{6}$  and so  $p^2 \mid \binom{3k}{k} \binom{6k}{3k}$  by the above. If  $k < \frac{p}{6}$ , then  $r \geq p-k > \frac{5p}{6}$  and so  $p^2 \mid \binom{3r}{r} \binom{6r}{3r}$  by the above.

Now putting all the above together we prove the lemma.

**Theorem 3.1.** *Let  $p$  be an odd prime and let  $x$  be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1-432x))^k \equiv \left( \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1-432x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=0}^k \binom{k}{r} (-432x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k}. \end{aligned}$$

Suppose  $p \leq m \leq 2p-2$  and  $0 \leq k \leq p-1$ . If  $k \geq \frac{2p}{3}$ , then  $2p \leq 3k < 3p$ ,  $6k \geq 4p$ ,  $p^3 \nmid (3k)!$ ,  $p^4 \mid (6k)!$  and so  $\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{(3k)!k!^3} \equiv 0 \pmod{p^2}$ . If  $\frac{p}{2} < k < \frac{2p}{3}$ , then  $3k < 2p$ ,  $6k > 3p$ ,  $p^2 \nmid (3k)!$  and  $p^3 \mid (6k)!$  and so  $\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{(3k)!k!^3} \equiv 0 \pmod{p^2}$ . If  $k < \frac{p}{2}$ , then  $m-k \geq p-k > k$  and so  $\binom{k}{m-k} = 0$ . Thus, from the above and Lemma 3.1

we deduce that

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1-432x))^k \\
& \equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k} \\
& = \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{3k}{k} \binom{6k}{3k} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)} \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{m=k}^{p-1} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)} x^{m-k} \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=0}^{p-1-k} \binom{3r}{r} \binom{6r}{3r} x^r \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \left( \sum_{r=0}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r - \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \right) \\
& = \left( \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \pmod{p^2}.
\end{aligned}$$

By Lemma 3.2 we have  $p^2 \mid \binom{3k}{k} \binom{6k}{3k} \binom{3r}{r} \binom{6r}{3r}$  for  $0 \leq k \leq p-1$  and  $p-k \leq r \leq p-1$ . Thus

$$\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

**Corollary 3.1.** *Let  $p$  be a prime greater than 3 and  $m \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv \left( \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} \left( \frac{1 - \sqrt{1 - 1728/m}}{864} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking  $x = \frac{1 - \sqrt{1 - 1728/m}}{864}$  in Theorem 3.1 we deduce the result.

**Lemma 3.3.** *Let  $p$  be a prime of the form  $4k+1$  and  $p = a^2 + b^2$  ( $a, b \in \mathbb{Z}$ ) with  $a \equiv 1 \pmod{4}$ . Then*

$$P_{[\frac{p}{6}]}(0) \equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]} \equiv \begin{cases} 2a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -2a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ 2b & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a-b. \end{cases}$$

Proof. By Lemma 2.1(i) and the proof of Theorem 2.1 we have

$$P_{[\frac{p}{6}]}(0) = \frac{1}{(-4)^{[\frac{p}{12}]}} \binom{[\frac{p}{6}]}{[\frac{p}{12}]} \equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]} \equiv (-3)^{-\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod{p}.$$

By Gauss' congruence ([BEW, p.269]),  $\left(\frac{p-1}{4}\right) \equiv 2a \pmod{p}$ . By [S1, Theorem 2.2],

$$(-3)^{\frac{p-1}{4}} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -1 \pmod{p} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ -\frac{b}{a} \equiv \frac{a}{b} \pmod{p} & \text{if } p \equiv 5 \pmod{12} \text{ and } b \equiv a \pmod{3}. \end{cases}$$

Thus the result follows.

Let  $p > 3$  be a prime. By the work of Mortenson[M] and Zhi-Wei Sun[Su3],

$$(3.3) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} \equiv \begin{cases} \left(\frac{p}{3}\right)(4a^2 - 2p) \pmod{p^2} & \text{if } p = a^2 + b^2 \equiv 1 \pmod{4} \text{ and } 2 \nmid a, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

In [Su1] Zhi-Wei Sun conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 7, 11 \pmod{12}, \\ (-1)^{\lfloor \frac{a}{6} \rfloor} (2a - \frac{p}{2a}) \pmod{p^2} & \text{if } 12 \mid p-1, p = a^2 + b^2 \text{ and } 4 \mid a-1, \\ \left(\frac{ab}{3}\right) (2b - \frac{p}{2b}) \pmod{p^2} & \text{if } 12 \mid p-5, p = a^2 + b^2 \text{ and } 4 \mid a-1. \end{cases}$$

In [Su2], Zhi-Wei Sun confirmed the conjecture in the case  $p \equiv 3 \pmod{4}$ .

Now we prove the above conjecture for primes  $p \equiv 1 \pmod{4}$ .

**Theorem 3.2.** *Let  $p$  be a prime of the form  $4k+1$  and so  $p = a^2 + b^2$  with  $a, b \in \mathbb{Z}$  and  $4 \mid a-1$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \begin{cases} 2a - \frac{p}{2a} \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -2a + \frac{p}{2a} \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ 2b - \frac{p}{2b} \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a-b. \end{cases}$$

Proof. From Lemma 3.3 we have  $P_{\lfloor \frac{p}{6} \rfloor}(0) \equiv 2r \pmod{p}$ , where

$$r = \begin{cases} a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ b & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a-b. \end{cases}$$

By the proof of Lemma 3.2 we have  $p \mid \binom{6k}{3k} \binom{3k}{k}$  for  $p > k > \frac{p}{6}$ . Thus, applying Lemma 2.3 and the above we get

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \sum_{k=0}^{\lfloor p/6 \rfloor} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv P_{\lfloor \frac{p}{6} \rfloor}(0) \equiv 2r \pmod{p}.$$

Set  $\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} = 2r + qp$ . Using Corollary 3.1 we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} \equiv \left( \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \right)^2 = (2r + qp)^2 \equiv 4r^2 + 4rqp \pmod{p^2}.$$

Thus, applying (3.3) we obtain  $\left(\frac{p}{3}\right)(4a^2 - 2p) \equiv 4r^2 + 4rqp \pmod{p^2}$ . Hence  $q \equiv -\frac{1}{2r} \pmod{p}$  and the proof is complete.

#### 4. Congruences for $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} / m^k$ .

**Theorem 4.1.** *Let  $p > 3$  be a prime,  $m \in R_p$ ,  $m \not\equiv 0 \pmod{p}$  and  $t = \sqrt{1 - 1728/m}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv P_{[\frac{p}{6}]}(t)^2 \equiv \left( \sum_{x=0}^{p-1} (x^3 - 3x + 2t)^{\frac{p-1}{2}} \right)^2 \pmod{p}.$$

Moreover, if  $P_{[\frac{p}{6}]}(t) \equiv 0 \pmod{p}$  or  $\sum_{x=0}^{p-1} (x^3 - 3x + 2t)^{\frac{p-1}{2}} \equiv 0 \pmod{p}$ , then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. Since  $\frac{1-t}{864}(1 - 432 \cdot \frac{1-t}{864}) = \frac{1}{m}$ , by Theorem 3.1 we have

$$(4.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv \left( \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} \left( \frac{1-t}{864} \right)^k \right)^2 \pmod{p^2}.$$

From the proof of Lemma 3.2 we know that  $p \mid \binom{3k}{k} \binom{6k}{3k}$  for  $[\frac{p}{6}] < k < p$ . Thus, using Lemma 2.3 and Theorem 2.3 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left( \frac{1-t}{864} \right)^k \equiv P_{[\frac{p}{6}]}(t) \equiv -\left( \frac{p}{3} \right) \sum_{x=0}^{p-1} (x^3 - 3x + 2t)^{\frac{p-1}{2}} \pmod{p}.$$

This together with (4.1) yields the result.

**Theorem 4.2.** *Let  $p > 3$  be a prime and  $m, n \in R_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\begin{aligned} \left( \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \right)^2 &\equiv \left( \frac{-3m}{p} \right) \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left( \frac{4m^3 + 27n^2}{12^3 \cdot 4m^3} \right)^k \\ &\equiv \left( \frac{-3m}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left( \frac{4m^3 + 27n^2}{12^3 \cdot 4m^3} \right)^k \pmod{p}. \end{aligned}$$

Moreover, if  $\sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) = 0$ , then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left( \frac{4m^3 + 27n^2}{12^3 \cdot 4m^3} \right)^k \equiv 0 \pmod{p^2}.$$

Proof. By the proof of Lemma 3.2 we have  $p \mid \binom{3k}{k} \binom{6k}{3k}$  for  $\frac{p}{6} < k < p$ . We first assume  $4m^3 + 27n^2 \equiv 0 \pmod{p}$ . As  $x^3 + mx + n \equiv (x - \frac{3n}{m})(x + \frac{3n}{2m})^2 \pmod{p}$  we see that

$$\begin{aligned} \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) &= \sum_{x=0}^{p-1} \left( \frac{(x - \frac{3n}{m})(x + \frac{3n}{2m})^2}{p} \right) = \sum_{\substack{x=0 \\ x \not\equiv -\frac{3n}{2m} \pmod{p}}}^{p-1} \left( \frac{x - \frac{3n}{m}}{p} \right) \\ &= \sum_{t=0}^{p-1} \left( \frac{t}{p} \right) - \left( \frac{-\frac{3n}{2m} - \frac{3n}{m}}{p} \right) = -\left( \frac{-2mn}{p} \right). \end{aligned}$$

Since  $m \not\equiv 0 \pmod{p}$  we have  $n \not\equiv 0 \pmod{p}$  and so  $\sum_{x=0}^{p-1} \left( \frac{x^3+mx+n}{p} \right) = -\left( \frac{-2mn}{p} \right) \neq 0$ . Thus the result holds in this case.

Now we assume  $4m^3 + 27n^2 \not\equiv 0 \pmod{p}$ . Set  $t = \frac{3n\sqrt{-3m}}{2m^2}$  and  $m_1 = \frac{1728 \cdot 4m^3}{4m^3 + 27n^2}$ . Then  $t = \sqrt{1 - \frac{1728}{m_1}}$ . From Theorems 2.1 and 4.1 we have

$$\left( \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) \right)^2 \equiv (-3m)^{\frac{p-1}{2}} P_{[\frac{p}{6}]}(t)^2 \equiv \left( \frac{-3m}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m_1^k} \pmod{p}.$$

If  $\sum_{x=0}^{p-1} \left( \frac{x^3+mx+n}{p} \right) = 0$ , using Theorems 2.1 and 4.1 we see that  $P_{[\frac{p}{6}]}(t) \equiv 0 \pmod{p}$  and so  $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m_1^k} \equiv 0 \pmod{p^2}$ . This completes the proof.

**Theorem 4.3 ([Su4, Conjecture 2.7]).** *Let  $p \neq 2, 11$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-32)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left( \frac{p}{11} \right) = -1, \\ \left( \frac{-2}{p} \right) x^2 \pmod{p} & \text{if } \left( \frac{p}{11} \right) = 1 \text{ and so } 4p = x^2 + 11y^2. \end{cases}$$

Proof. Taking  $m = -96 \cdot 11$  and  $n = 112 \cdot 11^2$  in Theorem 4.2 and then applying (2.8) we deduce the result.

**Theorem 4.4 ([Su4, Conjecture 2.8]).** *Let  $p \neq 2, 3, 19$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-96)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left( \frac{p}{19} \right) = -1, \\ \left( \frac{-6}{p} \right) x^2 \pmod{p} & \text{if } \left( \frac{p}{19} \right) = 1 \text{ and so } 4p = x^2 + 19y^2. \end{cases}$$

Proof. Taking  $m = -8 \cdot 19$  and  $n = 2 \cdot 19^2$  in Theorem 4.2 and then applying (2.9) we deduce the result.

**Theorem 4.5 ([Su4, Conjecture 2.9]).** *Let  $p \neq 2, 3, 5, 43$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-960)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left( \frac{p}{43} \right) = -1, \\ \left( \frac{p}{15} \right) x^2 \pmod{p} & \text{if } \left( \frac{p}{43} \right) = 1 \text{ and so } 4p = x^2 + 43y^2. \end{cases}$$

Proof. Taking  $m = -80 \cdot 43$  and  $n = 42 \cdot 43^2$  in Theorem 4.2 and then applying (2.9) we deduce the result.

**Theorem 4.6 ([Su4, Conjecture 2.9]).** *Let  $p$  be a prime such that  $p \neq 2, 3, 5, 11, 67$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-5280)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left( \frac{p}{67} \right) = -1, \\ \left( \frac{-330}{p} \right) x^2 \pmod{p} & \text{if } \left( \frac{p}{67} \right) = 1 \text{ and so } 4p = x^2 + 67y^2. \end{cases}$$

Proof. Taking  $m = -440 \cdot 67$  and  $n = 434 \cdot 67^2$  in Theorem 4.2 and then applying (2.9) we deduce the result.

**Theorem 4.7 ([Su4, Conjecture 2.10]).** *Let  $p$  be a prime with  $p \neq 2, 3, 5, 23, 29, 163$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-640320)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left(\frac{p}{163}\right) = -1, \\ \left(\frac{-10005}{p}\right) x^2 \pmod{p} & \text{if } \left(\frac{p}{163}\right) = 1 \text{ and so } 4p = x^2 + 163y^2. \end{cases}$$

Proof. Taking  $m = -80 \cdot 23 \cdot 29 \cdot 163$  and  $n = 14 \cdot 11 \cdot 19 \cdot 127 \cdot 163^2$  in Theorem 4.2 and then applying (2.9) we deduce the result.

**Theorem 4.8 ([S4, Conjecture 2.8]).** *Let  $p > 7$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-15)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ \left(\frac{p}{15}\right) 4C^2 \pmod{p} & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}. \end{cases}$$

Proof. Taking  $m = -35$  and  $n = 98$  in Theorem 4.2 and then applying (2.5) we deduce the result.

**Theorem 4.9 ([S4, Conjecture 2.9]).** *Let  $p > 7$  be a prime and  $p \neq 17$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{255^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ \left(\frac{p}{255}\right) 4C^2 \pmod{p} & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}. \end{cases}$$

Proof. Taking  $m = -595$  and  $n = 5586$  in Theorem 4.2 and then applying (2.7) we deduce the result.

**Theorem 4.10 ([S4, Conjecture 2.4]).** *Let  $p$  be a prime such that  $p \neq 2, 3, 11$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{66^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ \left(\frac{p}{33}\right) 4a^2 \pmod{p} & \text{if } p = a^2 + b^2 \equiv 1 \pmod{4} \text{ and } 2 \nmid a. \end{cases}$$

Proof. Taking  $m = -11$  and  $n = 14$  in Theorem 4.2 and then applying (2.2) we deduce the result.

**Theorem 4.11 ([S4, Conjecture 2.5]).** *Let  $p > 5$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{20^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}, \\ \left(\frac{-5}{p}\right) 4c^2 \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8}. \end{cases}$$

Proof. Taking  $m = -30$  and  $n = 56$  in Theorem 4.2 and then applying the result in the proof of Corollary 2.2 we deduce the result.

**Theorem 4.12 ([S4, Conjecture 2.6]).** *Let  $p > 5$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{54000^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \\ \left(\frac{p}{5}\right) 4A^2 \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}. \end{cases}$$

Proof. Taking  $m = -15$  and  $n = 22$  in Theorem 4.2 and then applying (2.3) we deduce the result.

**Theorem 4.13** ([S4, Conjecture 2.7]). *Let  $p > 5$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-12288000)^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 3 \mid p-2, \\ \left(\frac{10}{p}\right) L^2 \pmod{p} & \text{if } 3 \mid p-1 \text{ and so } 4p = L^2 + 27M^2. \end{cases}$$

Proof. Taking  $m = -120$  and  $n = 506$  in Theorem 4.2 and then applying (2.4) we deduce the result.

**Remark 4.1** From [O] we know that the only  $j$ -invariants of elliptic curves over rational field  $\mathbb{Q}$  with complex multiplication are given by

$$0, 12^3, -15^3, 20^3, -32^3, 2 \cdot 30^3, 66^3, -96^3, -3 \cdot 160^3, 255^3, -960^3, -5280^3, -640320^3,$$

coinciding with the values of  $m$  in (3.3) and Theorems 4.3-4.13.

## 5. Some conjectures on supercongruences.

**Conjecture 5.1.** *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{63k+8}{(-15)^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left(\frac{-15}{p}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{133k+8}{255^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left(\frac{-255}{p}\right) \pmod{p^2} \quad \text{for } p \neq 17, \\ \sum_{k=0}^{p-1} \frac{28k+3}{20^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 3p \left(\frac{-5}{p}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{63k+5}{66^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 5p \left(\frac{-33}{p}\right) \pmod{p^2} \quad \text{for } p \neq 11, \\ \sum_{k=0}^{p-1} \frac{11k+1}{54000^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv p \left(\frac{-15}{p}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{506k+31}{(-12288000)^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 31p \left(\frac{-30}{p}\right) \pmod{p^2}. \end{aligned}$$

Conjecture 5.1 is similar to some conjectures in [Su1].



**Conjecture 5.2.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{9n+4}{5^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4p \left(\frac{p}{5}\right) \pmod{p^2} \quad \text{for } p > 5, \\
\sum_{n=0}^{p-1} \frac{5n+2}{16^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2p \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{9n+2}{50^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2p \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p \neq 5, \\
\sum_{n=0}^{p-1} \frac{5n+1}{96^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv p \left(\frac{-2}{p}\right) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{6n+1}{320^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv p \left(\frac{p}{15}\right) \pmod{p^2} \quad \text{for } p \neq 5, \\
\sum_{n=0}^{p-1} \frac{90n+13}{896^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 13p \left(\frac{p}{7}\right) \pmod{p^2} \quad \text{for } p \neq 7, \\
\sum_{n=0}^{p-1} \frac{102n+11}{10400^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 11p \left(\frac{p}{39}\right) \pmod{p^2} \quad \text{for } p \neq 5, 13.
\end{aligned}$$

**Conjecture 5.3.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{3n+1}{(-16)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv p \left(\frac{-1}{p}\right) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{15n+4}{(-49)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4p \left(\frac{p}{3}\right) \pmod{p^2} \quad \text{for } p \neq 7, \\
\sum_{n=0}^{p-1} \frac{9n+2}{(-112)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2p \left(\frac{p}{7}\right) \pmod{p^2} \quad \text{for } p \neq 7, \\
\sum_{n=0}^{p-1} \frac{99n+17}{(-400)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 17p \left(\frac{-1}{p}\right) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{855n+109}{(-2704)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 109p \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p \neq 13, \\
\sum_{n=0}^{p-1} \frac{585n+58}{(-24304)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 58p \left(\frac{-31}{p}\right) \pmod{p^2} \quad \text{for } p \neq 7, 31.
\end{aligned}$$

For an integer  $m$  and odd prime  $p$  with  $p \nmid m$  let

$$Z_p(m) = \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{m^n} \sum_{k=0}^n \binom{n}{k}^3.$$

Then we have the following conjectures concerning  $Z_p(m)$  modulo  $p^2$ .

**Conjecture 5.4.** *Let  $p$  be an odd prime. Then*

$$Z_p(-16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid y, \\ 2p - 4x^2 \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid x - 3, \\ 4\left(\frac{xy}{3}\right)xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 5.5.** *Let  $p$  be an odd prime. Then*

$$Z_p(96) \equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

**Conjecture 5.6.** *Let  $p > 5$  be a prime. Then*

$$Z_p(-4) \equiv Z_p(50) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

**Conjecture 5.7.** *Let  $p > 5$  be a prime. Then*

$$Z_p(16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ and so } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ and so } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

**Conjecture 5.8.** *Let  $p > 3$  be a prime. Then*

$$Z_p(32) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and so } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

**Conjecture 5.9.** *Let  $p > 7$  be a prime. Then*

$$Z_p(5) \equiv Z_p(-49) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 15y^2 \equiv 1, 19 \pmod{30}, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 5y^2 \equiv 17, 23 \pmod{30}, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 29 \pmod{30}. \end{cases}$$

**Conjecture 5.10.** Let  $b \in \{7, 11, 19, 31, 59\}$  and let  $f(b) = -112, -400, -2704, -24304, -1123600$  according as  $b = 7, 11, 19, 31, 59$ . If  $p$  is a prime with  $p \neq 2, 3, b$  and  $p \nmid f(b)$ , then

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + by^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } 2p = x^2 + 3by^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } 2p = 3x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-3b}{p}\right) = -1. \end{cases}$$

**Conjecture 5.11.** Let  $b \in \{5, 7, 13, 17\}$  and  $f(b) = 320, 896, 10400, 39200$  according as  $b = 5, 7, 13, 17$ . If  $p$  is a prime with  $p \neq 2, 3, b$  and  $p \nmid f(b)$ , then

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6by^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p = 2x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 2by^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } p = 6x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6b}{p}\right) = -1. \end{cases}$$

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